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Calculating Arc Parameters

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Tim Olsen recently discovered that I'm a mathematician, and asked me to write a short sidebar incorporating and extending information about circular arcs provided in several articles which have appeared in American Lutherie. The central question he asked me to address was: if I know the width of my workboard (or brace or top plate) and I know how much higher the edges are than the middle, what is the radius of that arc? In the following are formulas that give an answer to that question, and to several other related ones. In fact, we'll find a general relationship between three quantities: the span of an arc, the center deflection of the arc, and its radius. Then given any two of these quantities, we'll be able to find the third.

Our basic approach will be an analytical one, using the Pythagorean Theorem to develop an algebraic equation which relates the three quantities. Then we'll solve for each quantity in turn, producing three formulas which will allow us to find each of the quantities if we know the other two. This approach is similar to that used by Chris Foss in American Lutherie #9 (p. 58) and described by Phil Kearny in a communication to the GAL; Maryann Faller of Adirondack Community College suggested an elegant geometric method using similar triangles. In mathematics, there are often many different ways to saw the same log - that's what makes it so interesting (or frustrating, depending on your particular taste). After producing the necessary formulas, we'll illustrate each with a sample lutherie application.

Finally, I've included a short discussion about other types of curves which are useful and appear in applications.

Formulas

In the accompanying diagram (figure 1), L represents the span of the arc, D represents its center deflection, and R represents its radius. The fundamental relationship between these comes from applying the Pythagorean Theorem to the right triangle in the figure; this gives

$$(R - D)^2 + (L/2)^2 = R^2$$

This is the fundamental relationship between the span, deflection and radius. Now all we have to do is use some algebra to solve for each variable in terms of the other two. The resulting formulas and some applications are given below; the derivations of the formulas are given in the appendix.

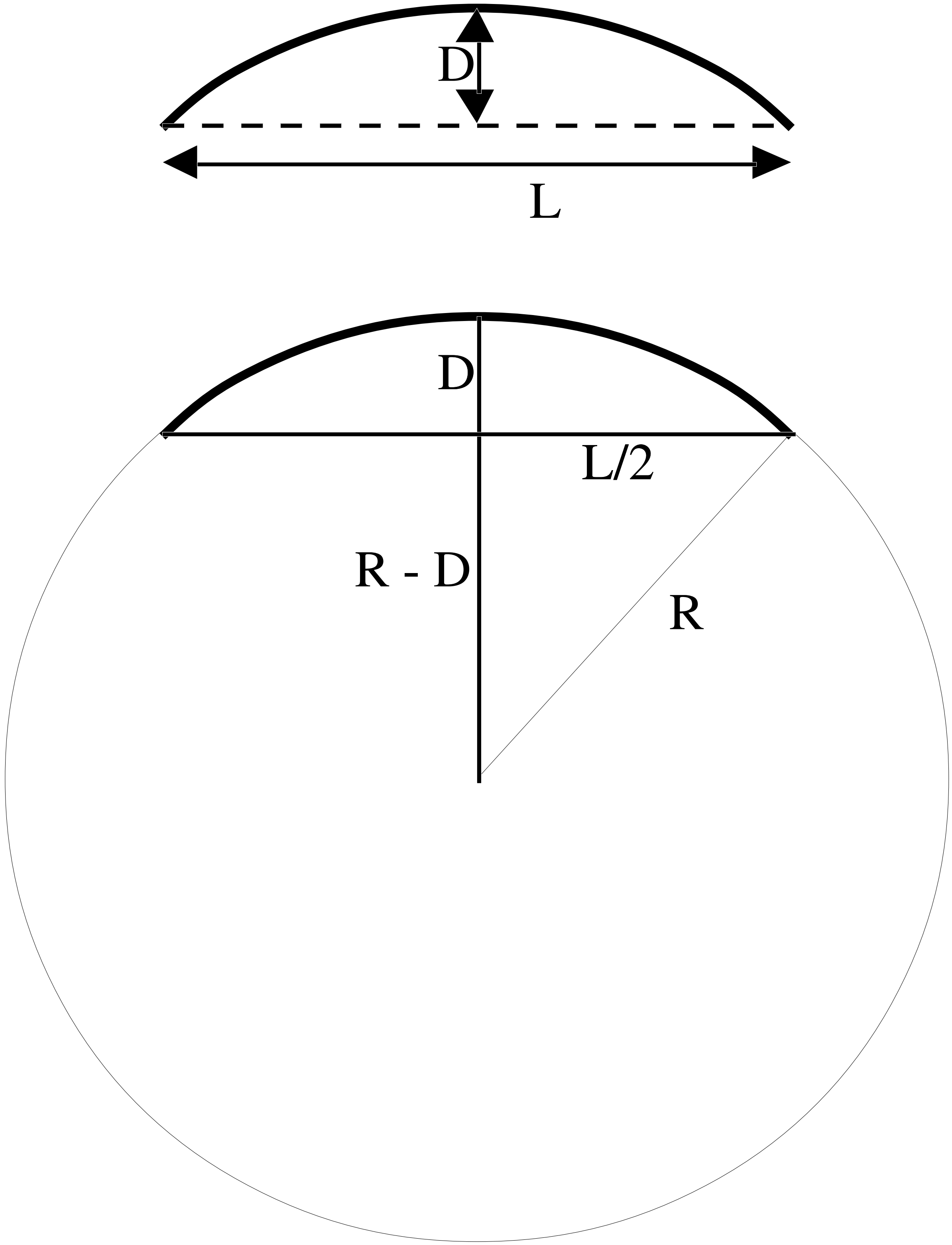

Figure 1

Solving for R in terms of D and L:

Result:

$$R = \frac{D^2 + (L/2)^2}{2D}$$

Application: In their book Guitarmaking, Tradition and Technology, Bill Cumpiano and Jon Natelson specify that the arching for the back braces for the the steel-string guitar being built should have a 1/4" center deflection (they call it an offset) in a 16" span (p. 150); the radius that this corresponds to can be calculated using the above formula. Here, D = .25" and L = 16", giving L/2 = 8" and thus

$$R = \frac{(.25)^2 + (8)^2}{2 \cdot (.25)} = \frac{.0625 + 64}{.5} = \frac{64.0625}{.5} = 128.125 \quad \text{inches,}$$

or the radius is R = 10 ft. 8 in. (ignoring the last .125 inch).

A Handy Approximation

If the center deflection D is much smaller than the span L, the D^2 term in the formula above will be insignificant compared to the $(L/2)^2$ term, and can be neglected; this leads to a simplified version of the formula:

If L is at least 20 times as large as D, then the radius is given approximately by

$$R = \frac{L^2}{8D} \quad ,$$

and the approximation will be good to within 1% of the true radius.

If we use this simpler formula on the example above, it gives

$$R = \frac{16^2}{8 \cdot (.25)} = \frac{256}{2} = 128 \quad \text{inches,}$$

which is plenty close enough to the exact value of 128.125 inches found above.

Solving for D in terms of R and L:

Result:

$$D = R - \sqrt{(R^2 - (L/2)^2)}$$

Application: In his book The Steel String Guitar: Construction and Repair, David Russell Young specifies that the braces for the top of the guitar be arched with a 25 foot radius. Suppose we want to make a template to use to check the arching of the top and scribe the arch onto the sides of the braces; if the template will be 24" long, how much higher should the center be than the ends?

Here, we know the radius R = 25 * 12 = 300 inches, and the span is L = 24 inches; note that it's important to use the same units for all of the quantities in the formula (don't mix inches and feet, or centimeters and meters). Then the above formula will give us the needed deflection D:

$$D = 300 - \sqrt{(300^2 - (24/2)^2)} = 300 - \sqrt{(300^2 - 12^2)} = 300 - \sqrt{(90000 - 144)}$$, i.e.,

$$D = 300 - \sqrt{(89856)} = 300 - 299.7599 = .2401,$$

or D = .24 inches.

Thus the deflection would be very nearly 1/4" for this span of 24", i.e., the braces should be 1/4” higher in the middle than at the ends.

Solving for L in terms of R and D:

Result:

$$L = 2 \cdot \sqrt{(2 \cdot R \cdot D - D^2)}$$

Application: I made a fingerboard radius block to put a 10"-radius crown on my fretboards. To construct it, I laminated several layers of thin plywood, spreading glue on the mating faces and then clamping them into the appropriate arch until the glue dried. To get the correct arch, I placed two 1/4" dowels on my workbench, several inches apart and parallel, laid the plywood-glue sandwich on top, and then clamped the plies down to the workbench surface with a caul halfway between the two support dowels. Thus the plies took on an arch that had a center deflection of 1/4"; but I needed to know how far apart the dowels on the workbench should be placed to get the desired 10" radius. This amounted to solving for the span L in an arc when the deflection and radius were known. Thus I knew R = 10 inches and D = .25 inches; the formula above gave me the span L:

$$L = 2 \cdot \sqrt{(2(10)(.25) - (.25)^2)} = 2 \cdot \sqrt{(5 - .0625)} = 2 \cdot (2.222) = 4.444$$

or L = 4.44 inches. Thus I placed the dowels 4 1/2" apart on the workbench. (I just wanted a radius of approximately 10"; thus I didn't worry about the difference between 4.44" and 4.5". The previous formulas indicate that if the span is 4.5" and the center deflection is .25", then the actual radius comes out to be 10.25", which was close enough for me. In fact, this gives the radius of the outside of the sanding block (the convex surface); I'd be using the inside (concave) face to crown the fretboard. Since the thickness of the block was 3/4", the actual crown radius I obtained was 10.25" - .75" = 9.5"; again, this was fine for me.

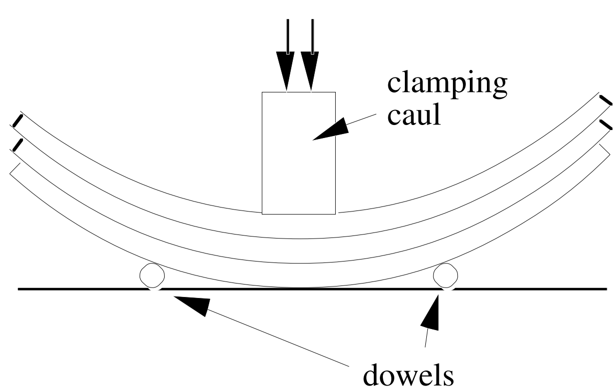

Figure 2

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Sketching Arcs

A convenient and simply made device for sketching an arc of a circle with a large radius, which incidentally uses the span and center deflection of the arc, is the "long compass". This device consists of two boards nailed together to form a slight angle, and held against two brads hammered into the workpiece. A pencil is then placed in the vertex; as the assembly is slid along the brads, the pencil traces out a circular arc (figure 3). The above device, or a variation on this theme, has been described a number of times in the woodworking and lutherie literature. For descriptions, see the GAL publication Lutherie Tools (p. 90), or Fine Woodworking, issues 28 (p. 14), 57 (p. 8), and 59 (p. 4).

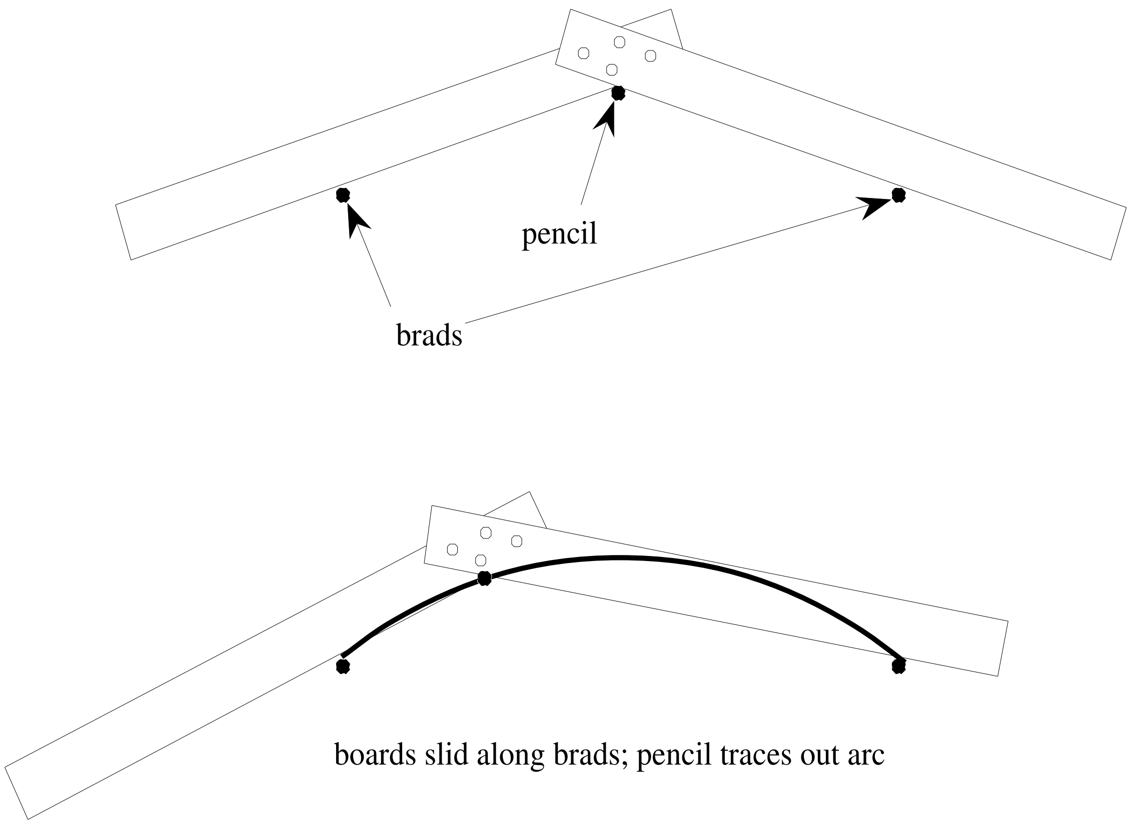

Figure 3

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Other Curves in Lutherie

The above gives a luthier tools to use to calculate the parameters of any desired circular arc. However, there are other curves that arise naturally that can be very useful in lutherie. First, while the "long compass" mentioned above is a great tool for sketching an arc of a circle with large radius, it requires hammering a couple of brads into the piece being marked for the compass to ride on. This suggests an even easier method for sketching the arc: put a thin, narrow piece of wood against the brads, flex it so it passes through the desired center point of the arc, and run a pencil along it to sketch a smooth curve that passes through all three points. This method has in fact been used for centuries; the thin piece of wood used is called a draftsman's spline, and the resulting smooth curve is called a spline curve. In fact, by hammering in a number of brads at desired spots, this method can be used to create smooth serpentine curves that pass through any specified number of points. So why not use the spline to sketch our circular arc? Well, the problem is that pieces of wood (and other materials) don't bend in circular arcs when they're deflected: the curve sketched will be slightly "peaked" in the center when compared with a circular arc drawn with the long compass (try it and see). (Mathematically, these spline curves (for relatively small deflections) are parametric piecewise-cubic polynomial curves with maximum continuity at the segment junctions; you can see why they're called "splines" for short.)

This isn't to say that you shouldn't use splines instead of circles; just recognize that they're not the same thing! In some sense, spline curves are more "natural", since they're the shape physical beams (like braces) naturally take on when they're deflected by some applied force (a discussion of this can be found in books on Mechanics of Materials, like the one by James Gere and Stephen Timoshenko). Also, the smooth curves produced in most drawing or drafting computer software packages are splines (they're sometimes also called Bezier curves or B-splines) or an extension called NURBs (non-uniform rational B-splines). In these applications, rather than specifying points that the curve passes through, the designer works with "control points" that pull the curve toward (but not through) them, in the way that magnets would pull a steel guitar string. I use spline curves in computer software when I'm designing a new guitar body style. They produce smooth curves which permit very fine adjustment and allow a great number of body shapes to be explored. The only thing that's a bit unusual about working with splines from the lutherie perspective is that the curves smoothly blend from sharp turns (say at the waist of a guitar body) to broad turns (at the lower bout); there's no single place where the radius of the curve suddenly changes from one value to another, as would be the case if the body shape were designed with circular arcs. Thus hot-pipe bending the sides of a spline-designed body shape requires a little getting used to, since the radius of the curve is continually changing. But then again, this makes for smoother transitions along the body outline.

References:

Guitarmaking Tradition and Technology, Bill Cumpiano and Jon Natelson, Rosewood Press, Amherst, MA, 1987.
The Steel String Guitar: Construction and Repair, David Russell Young, The Bold Strummer, Ltd., Westport, CT, 1987.
Lutherie Tools, Tim Olsen, Cyndy Burton, eds., Guild of American Luthiers, Tacoma, WA, 1990.
Mechanics of Materials, 4th edition, J. Gere, S. Timoshenko, PWS Publishing Co., 1997

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Appendix: Derivations of formulas

1. Solving for R in terms of D and L:

Result:

$$R = \frac{D^2 + (L/2)^2}{2D}$$

Derivation:

$(R - D)^2 + (L/2)^2 = R^2$

expand the $(R - D)^2$ term (using FOIL - bring back bad memories?) to get

$R^2 - 2RD + D^2 + (L/2)^2 = R^2$,

subtract R^2 from both sides to get

$-2RD + D^2 + (L/2)^2 = 0$,

add 2RD to both sides to get this term on the right,

$D^2 + (L/2)^2 = 2RD$,

then divide both sides by 2D to isolate the R on the right, giving our result:

$= R$.

2. Solving for D in terms of R and L:

Result:

$$D = R - \sqrt{(R^2 - (L/2)^2)}$$

Derivation:

$(R - D)^2 + (L/2)^2 = R^2$

subtract $(L/2)^2$ from both sides, giving

$(R - D)^2 = R^2 - (L/2)^2$.

Then take the square root of both sides to get

$$R - D = \sqrt{(R^2 - (L/2)^2)} \quad .$$

Finally, subtract R from both sides to isolate the D,

$$-D = -R + \sqrt{(R^2 - (L/2)^2)} \quad ,$$

then negate both sides to get rid of the negative sign in front of D, giving the result.

3. Solving for L in terms of R and D:

Result:

$$L = 2 \cdot \sqrt{(2 \cdot R \cdot D - D^2)}$$

Derivation:

$(R - D)^2 + (L/2)^2 = R^2$

Subtract $(R - D)^2$ from both sides to get

$(L/2)^2 = R^2 - (R - D)^2$.

Expanding the $(R - D)^2$ term on the right gives

$(L/2)^2 = R^2 - (R^2 - 2RD + D^2)$,

and simplifying the right leaves

$(L/2)^2 = 2RD - D^2$.

Now take the square root of both sides to get

$$L/2 = \sqrt{(2\,RD - D^2)} \quad ;$$

then multiplying by 2 gives the desired result: