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What Happens if I Make It Bigger? Estimation in the Workshop

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I was recently gluing top braces for a steel-string acoustic guitar, and through sheer laziness was led to the question: how much do small changes in dimensions used in elements of our guitars affect their function? To be more concrete, I started wondering this because I was gluing on the small support braces that run from the X-brace to the edge of the top to provide extra support against the tension of the strings on the bridge, and didn't have any braces made up of exactly the right size (cross-section). For the model of instrument I was constructing, I use braces that are 1/4" wide and 7/16" high for these small support braces, but the only brace stock I had at hand was 5/16" wide by 3/8" high. To make up new brace stock is (a tiny bit) less trivial than it might seem, since I use a top that is slightly arched and consequently arch all of the gluing surfaces of the braces. (I use a jig that I read about some time back in American Lutherie, in which a brace is flexed and clamped into a curved (bent) configuration and then planed flat; when released from the jig, the brace springs back to straighten itself out, and the flat-planed surface becomes a beautiful smooth convex curve matching the radius of the top. There's a picture of the jig in an article on arched workboards that's posted on my web page.) Anyway, I confess that just making a few new braces with the correct profile would have been a pretty simple matter, but I wondered nonetheless about the difference it would make if I used the existing brace stock in place of the desired stuff. My principle focus was on stiffness: how much stiffer or less stiff would a 5/16" wide by 3/8" high brace be than a 1/4" wide by 7/16" high brace?

I wasn't interested in exact values of stiffness, in, say, units of pounds of force needed per inch of center deflection. All I wanted to know was relative change: whatever the stiffness of the original (desired) brace, what percentage stiffer (or less stiff) would the new (existing) brace be? Interestingly enough, this doesn't require knowledge of any of the material properties of the brace stock (e.g., Young's modulus), ASSUMING that the two braces are made of exactly the same stock. This is actually a trickier issue than it might seem, since the material properties of spruce vary widely from tree to tree, and even from place to place within the same log! In fact, some luthiers employ a jig to measure the stiffness of each brace to compensate for this lack of uniformity. But I'm going to sweep this issue under the rug; let's just assume that you have a piece of rough stock, and have the choice of what dimensions you'll surface it to. In addition, I wanted a fairly easy-to-apply criterion - a rule of thumb - that would give a decent approximation to the change in stiffness brought about by a change in dimensions. I didn't want to have to run to the computer to fire up a spreadsheet to calculate the difference; I wanted to be able to say to myself, "If the brace height is increased by 5%, its stiffness is increased by 15%," or whatever.

The desired approach comes from differential calculus. (Please don't turn the page immediately - that's the last time I'll use that word, I promise!) A beautiful result from the subject is the following:

if quantity b is related to quantity a by a formula of the form

$$b = C\,a^m,$$

where C and m are constants, then a 1% change in a will cause approximately an m% change in b.

In the above, we're assuming that b is related to a by a power formula: that a is raised to some power (here it's m) to get b. The factor C is just a proportionality constant; in fact, the above relationship is often expressed as, "b is proportional to a raised to the power m," and written

$$b \;\alpha\; a^m \;,$$

ignoring the proportionality constant. It's useful to do this since the proportionality constant doesn't affect relative changes: the percent change in b depends only on the percent change in a and the power m. In fact, another way to express the result is: the percent change in b will be approximately m times the percent change in a.

It's important to note that this approximation is generally accurate only for small percentage changes in a, up to, say, 20%. (It's accurate over a wider range when the power m is close to 1, and accurate over a narrower range when m is substantially larger than 1.)

Let's see how this can be applied in the brace scenario. The relationship between the stiffness S of a brace (as measured by the amount of applied force needed to achieve some specified deflection at the midpoint of the brace) and its length L, width w and height h, is given by the proportionality relation (Timoshenko, p. 146):

$$S \;\alpha\; \frac{w \cdot h^3}{L^3} \;=\; w^1 \cdot h^3 \cdot L^{-3}$$

(Recall that w is the same as w raised to the power 1, and that the power of 3 in the denominator is the same as having a power of -3 in the numerator.) We say that the stiffness of the brace is proportional to the width w of the brace, the height h of the brace cubed, and is inversely proportional to the length L cubed. Using this, we can calculate how changes in the dimensions affect the stiffness. For example, a 1% increase in the width of the brace will bring about a 1% increase in the stiffness, since the power m is 1. Similarly, a 1% increase in the height will cause a 3% increase in stiffness (m = 3), and a 1% increase in length will cause a -3% change in stiffness (m = -3), i.e., will cause a 3% decrease in stiffness. This makes sense qualitatively: increasing the width or height of a brace should increase the stiffness, while increasing its length should decrease stiffness. (A longer beam is more flexible at the center; think of placing your weight at the center of a 1-foot-long 2x4 supported at its ends on sawhorses, versus standing at the center of a 12-foot-long 2x4 similarly supported; the first I would do, the second I wouldn't recommend!) What's more interesting is the quantitative interpretation: while increases in width and height both increase the stiffness, changes in height have a much greater effect than changes in width - three times the effect, in fact.

Now let's consider the actual values originally discussed. Starting with the desired brace dimensions of 1/4" wide by 7/16" high, the new dimensions to be considered are 5/16" wide by 3/8" high. To get the relative changes, divide the amount of change by the original value. For the width, the change is 1/16"; dividing by 1/4" gives .25, for a percent change of 25%. Thus by increasing the width by 25%, we'd expect to increase the stiffness by 25% (since the power $m = 1$ for the width). (Actually, the percentage change in width here is at the limit of what we should use for the approximation to be accurate; however, when the power $m = 1$, the method actually works for <u>any</u> percent change, however large.) For the height, the change is 1/16"; dividing by the original height of 7/16" gives .15, for a percent change of 15%. Note, however, that this is a 15% <u>decrease</u> in height. Since the power in the formula is $m = 3$ for the height h, the percent change in height will cause a percent change of 3 times 15%, or 45%, in the stiffness. Thus the stiffness will be decreased by approximately 45% due to the 15% reduction in height. The two effects can be combined: the 25% increase in stiffness due to the greater width combined with the 45% decrease due to reduced height gives an overall 20% decrease in stiffness. Again, this is just an approximation; an exact calculation using the formula shows the actual change to be a 21.3% decrease. But the approximation is certainly close enough to be useful in the shop. Note that the length didn't enter into the calculation, since it's assumed that it would be the same for the two braces and hence doesn't change. It's also worth noting that this doesn't in any way take into account the tapering of the braces, either at the ends or in cross-section. These effects would be very significant in determining the actual stiffness of the braces. This approach just gives a quick approach to determining the relative initial stiffness before any shaping has occurred.

It's clear that for this to be of any use, one needs to know the formula giving the dependence of the quantity of interest on the dimensions being varied. In the following are some specific settings and relevant formulas which might arise in practice.

Fabrication tolerances

The above method is especially well suited to cases where the changes in dimension aren't due to deliberate design changes (or laziness), as in the above, but are due instead to the tolerances that are naturally present in any shaping operation - for example, what if braces are made 1/2" high, plus or minus 1/64"? Since tolerances are usually small, the percent changes will usually be small, and thus will be well within the 20% limit for the method to be reasonably accurate. This approach can help to determine the variation in stiffness caused by such dimensional variation, and help to evaluate whether a particular tolerance is sufficient to guarantee a specified maximum variation in the quantity of interest. In the above example, the change in height is (plus or minus) 1/64"; when divided by the target value of 1/2", this gives .03, or a percent variation of plus or minus 3% in height. Since the power m = 3 in the formula above, this gives a variation of 3 times 3%, or 9%, in stiffness. Thus the stiffness will be within plus or minus 9% of the target value if the braces are made with this tolerance.

Suppose, though, that you desire the stiffness to vary by no more than 5% from the target value. What should the dimensional tolerance for the height be held to to achieve this? Well, since the power is 3 in the formula, each 1% change in height will cause a 3% change in stiffness; so the height should be allowed to vary no more than 5% divided by 3, or 1.66% Thus we want the tolerance divided by the height to be no more than .0166, i.e., $\frac{T}{(1/2)} = .0166$, or $T = .0166 * (1/2") = .008"$. Thus to keep the stiffness within 5%, you'd need to keep the brace tolerance to within about one hundredth of an inch. However, the variation in stiffness due to this tolerance would most likely be exceeded by variation in the material stiffness of spruce from piece to piece, so it's probably not worthwhile to agonize over dimensional tolerances of this order (though it's interesting to see the effects they can have). In fact, this argues for measuring the stiffness of each brace directly, since this will take into account both dimensional tolerances and material variations.

Top plate thickness and stiffness

A related issue is top plate stiffness as a function of its thickness. There are actually two sides to this matter for luthiers: the stiffness of the plate due to static loads, such as the pull of the strings on the top due to their tension, and the dynamic responsiveness of the top, i.e., its tendency to want to vibrate when a varying force is applied. We'll look at the first matter here, and the second in the next section.

The stiffness of a plate, i.e., its resistance to deflection for an applied load, is a very complicated matter that depends on the shape of the plate and how its is supported at its edge (e.g., is it simply supported, just laying flat on a "knife edge" at its perimeter, or is it clamped at the edge) as well as its thickness and material properties. Indeed, the determination of the deflection that will be caused by a specified applied force requires the solution of a partial differential equation that can be explicitly solved only in simple special cases (such as the case of a rectangular or circular plate). Add to this the complicated array of bracework on the underside of a typical guitar top, and it becomes virtually impossible to find any algebraic formula that will give the dependence of stiffness on the values of all of these quantities. However, ignoring the braces and considering just plate thickness t, it is possible to arrive at a formula for the stiffness S; assuming all other quantities (plate shape, support, material properties) remain the same, we have

$$S \;\alpha\; t^3 \qquad \text{(Den Hartog, Chapter 4, "Bending of Flat Plates")}$$

This relationship is the same as that for brace stiffness and height considered above, with an exponent of m = 3. Thus every 1% change in plate thickness will cause a 3% change in stiffness (resistance to static deformation).

String tension versus diameter, length and pitch

The tension need to bring a string up to pitch depends on its diameter, length and the desired pitch: the larger the diameter, the longer the scale length, or the higher the pitch, the higher the tension needed. The exact relationship between these quantities is (Kinsler and Frey, p. 47):

$$T \;\alpha\; d^2 \cdot L^2 \cdot f^2$$

(where *T* is tension, *d* is diameter, *L* is length and *f* is frequency).Since the exponents are all equal to 2, the effect of any percentage change in any of the diameter, length or pitch is to cause double the percentage change in the string tension.

For example, if a .012" diameter E string is changed to a .013" diameter string (while the pitch and scale length are constant), the percent change in diameter is 8% (.001 divided by .012 times 100), which will effect a 16% increase in string tension for this string. Similarly, if scale length is increased from 25.4" to 26", the percent change is 2.4%, resulting in about a 5% increase in overall string tension for the same gauge strings, which places greater stress on the top and neck of the instrument. Finally, the change from a concert pitch of A422.5 Hz, as found on a tuning fork used at one point by Handel (Wikipedia), to the standard concert pitch of A440 used today, requires stringed instruments to be tuned higher and thus increases the stress on the instruments. The change represents a 4% increase in frequency; this then corresponds to an 8% increase in string tension.

Frequency versus fret position

As a final application, consider the variation in string pitch due to the finite tolerance in fret location. When I'm scribing fret positions on a fingerboard, I use a ruler graduated in hundredths of an inch; thus the position of any fret should be accurate to within plus or minus .005" (half a hundredth). What fractional variation does this imply in the frequency of the fretted note? The previous formula applies here; when rearranged to express frequency f in terms of string length *L* (now the fretted length), and ignoring the other two quantities, we get

$$f \;\alpha\; \frac{1}{L} \;=\; L^{-1}$$

This is an inverse relationship, indicated by the fact that *L* is in the denominator, or, equivalently, by the fact that its exponent *m* = -1 when rearranged. This implies that an increase in length will produce a decrease in frequency, as expected: the longer the string, the lower the tone, all other things being equal. The exponent of -1 indicates that a 1% increase in length will produce an equal but opposite 1% decrease in frequency.

So consider my scale length of 25.5", and look at the 12th fret. The intended distance from the saddle to the fret is 12.75", but my tolerance is plus or minus .005". The relative change in length produced by this tolerance is only plus or minus .04 percent (note that this has already been multiplied by 100 to get percentage; the percent change is really small)! This will then cause an equal percent change in the frequency of the fretted note. The question arising is, will this be audibly significant? Looking at the high E string, tuned to the standard frequency of 329.6 Hz, the intended value of the frequency of the note at this fret is exactly twice this, or 659.2. If this is increased by .04%, the frequency will increase by .26 Hz, becoming 659.46 Hz. The notes differ by about 1/4 Hz; thus if played together, they would generate a beat (due to the close but differing frequencies) which would "pulse" once every four seconds. Due to the natural decay of the notes, this would probably not be noticeable; in addition, the change in frequency just caused by fretting the string (stretching the string as it's pushed against the fingerboard) would likely cause as much of a difference, if not more.

Conclusion

To sum up, formulas relating quantities of interest to those we ordinarily measure can provide us with some simple but useful heuristics for estimating changes.

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